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Analytical solutions for turbulent non-Boussinesq plumes

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Analytical solutions are developed for non-Boussinesq turbulent plumes rising from horizontal area sources in unconfined quiescent environments of uniform density. The approach adopted follows and extends an earlier approach for Boussinesq plumes and replaces the non-Boussinesq area source of interest and located at $z = 0$ with an idealized point source located at a virtual origin $z = z_v$ such that the flow above the idealized source approximates that from the actual source. Asymptotic analytical expressions are developed for the location of the virtual source that are valid for large vertical distances above the non-Boussinesq source. The non-Boussinesq source is characterized by a non-dimensional parameter Γ_{nb} which is a measure of the relative strengths of the mass, momentum and density deficit fluxes at, or at a specified height above, the source. The vertical distance between the actual and virtual sources scales on the length scale ℓ that characterizes the height over which the flow is non-Boussinesq and expressions for z_v/ℓ are developed for lazy ($\Gamma_{nb} > 1$) and forced plume ($\Gamma_{nb} < 1$) sources. For pure-plume source conditions $\Gamma_{nb} = 1$, and the virtual source provides an exact representation of the actual plume above $z = 0$. The limiting cases of a nearly pure lazy plume and of a highly lazy plume are also explored analytically. For fire plumes, Γ_{nb} is determined from the balance of fluxes immediately above the combustion region and a procedure for estimating these fluxes is given. Solutions expressing the dependence of the mass flux with height are also developed for the near-field flow regions and thereafter an approximation for the mass and momentum fluxes valid for all heights and for source conditions yielding $0 < \Gamma_{nb} < \infty$ is deduced. Applications of the model may include plumes above fires and forced releases of highly buoyant gas into the atmosphere, for example, following the rupturing of a pressurized container vessel.

1. Introduction

Plumes are classed as non-Boussinesq at their source if the difference in density between the fluid rising from the source and the surrounding environment is significant compared with a typical reference density ρ_0, for example, the density of the local surroundings. A discussion of this classification is given, for example, by Spiegel & Veronis (1960). Examples of non-Boussinesq plumes rising from area sources are widespread in nature, e.g. plumes of hot gases rising above fires, and in industry, e.g. plumes of highly buoyant gas released following the localized rupturing of a pressurized container. As a turbulent plume rises, it is diluted owing to entrainment of fluid from the surrounding environment and the density contrast decreases with height. Above a characteristic vertical height ℓ (Woods 1997) the density contrast is small compared with ρ_0 and the Boussinesq approximation may be applied. The classical theory of Boussinesq plumes has been established by Morton, Taylor & Turner (1959, hereinafter referred to as MTT).

Non-Boussinesq plumes, and in particular fire plumes, have been the subject of numerous theoretical, laboratory and field studies and a detailed review of the dynamics of fire plumes is given by Drysdale (1985). Rooney & Linden (1996, hereinafter referred to as RL) extend the classical Boussinesq plume analysis of MTT to the case of rising non-Boussinesq plumes from point sources. RL develop conservation equations based on the assumptions of self-similar 'top-hat' profiles for density deficit and vertical velocity in the plume. Using their similarity solutions, they show that the entrainment coefficient, which models the horizontal inflow into the plume, is a function both of the local vertical velocity in the plume and of the local density contrast. Their analytical solutions of the resulting plume conservation equations yield power-law relationships for the mass flow rate and the density deficit as a function of the distance above the source. These solutions are discussed further in § 2.2.

In many cases of practical interest, plumes rise from sources of finite area, rather than points, and may include non-zero source (i.e. at $z = 0$, z denoting the vertical coordinate measured from the source) fluxes of mass and momentum. For simplicity, we shall use the notation (B_0, M_0, Q_0) for this general area source, where B_0, M_0 and Q_0 denote the buoyancy, momentum and volume fluxes at the source, respectively. In the case of fire plumes, these fluxes are not typically known at the source, but may be estimated at some height above it (see § 5.5). It is convenient therefore to classify the source in terms of the fluxes defined at a specified height $z = z_h$ (> 0) above the source and the notation is then (B_h, M_h, Q_h). For an axisymmetric plume, the actual fluxes of volume Q, buoyancy B and momentum M are defined as

$$Q = 2\pi \int_0^\infty wr \, \mathrm{d}r, \tag{1.1}$$

$$B = 2\pi \int_0^\infty g \frac{\rho_e - \rho}{\rho_e} wr \, \mathrm{d}r, \tag{1.2}$$

$$M = 2\pi \int_0^\infty \rho w^2 r \, \mathrm{d}r, \tag{1.3}$$

respectively, where r denotes the radial or cross-plume coordinate, $w(r, z)$ the vertical velocity, and $\rho(r, z)$ and ρ_e denote the density of the plume fluid and the environment, respectively. In some applications, for example the study of volcanic plumes which may extend to the stratosphere, the background density is not constant and $\rho_e = \rho_e(z)$. In other applications, for example the study of fires, it is often a reasonable assumption that the background density is constant. In the present paper, we restrict ourselves to a constant ambient density $\rho_e = \rho_0$.

For Boussinesq plumes, a number of approaches, both theoretical and based on the results of laboratory experiments, are available to model the flow above general area sources; the theoretical approaches have a common thread as they are based on relating the flow above the actual source to the flow above an appropriately located point source – a virtual source located at a virtual origin. The concept behind a virtual origin is to replace the actual source $(B_0, M_0, Q_0, z = 0)$ with a point source of buoyancy alone $(B_0, 0, 0, z = z_v)$ such that the flow from the two plumes match above the actual source. The plumes match asymptotically only in the limit of large z, unless the actual plume satisfies the conditions of a pure plume at the source (see

§ 3). In this case, flows above virtual and actual sources are identical for all $z \geqslant 0$. The vertical offset between the two origins is referred to as the virtual-origin correction. Once the virtual-origin location is known, the theory for a point-source plume may be applied directly from that point.

A summary of these approaches is given by Hunt & Kaye (2001). Similar empirical approaches have been identified and applied to non-Boussinesq fire plumes, see for example Drysdale (1985) and Heskestad (1998).

The aims of this paper are to develop a theoretical virtual-origin correction for non-Boussinesq plumes, thereby generalizing the results for Boussinesq plumes (see Morton 1959; Hunt & Kaye 2001), and to develop matched expansions valid both in the far field above the plume (as a virtual-origin correction is) and close to the source.

The paper is laid out as follows. In § 2, the equations of motion governing the flow in a rising turbulent non-Boussinesq plume from an axisymmetric source in a quiescent and uniform environment are presented together with the assumptions on which they are based. The length scale ℓ that characterizes the flow and the dimensionless parameter Γ_{nb}, a measure of the relative strengths of the mass, momentum and density deficit fluxes and the non-Boussinesq equivalent of the parameter Γ_b for Boussinesq flows, are then introduced in § 3, together with an analytical expression for the position of the virtual origin for pure plumes. Section 4 is devoted to lazy plumes and § 5 to forced plumes. In these two sections, we derive the virtual-origin correction in two steps and then propose matched asymptotic expansion solutions of the plume equations. The conclusions are drawn in § 6.

2. Non-Boussinesq plumes

2.1. *Conservation equations*

We let $\widetilde{w}(r,z)$ and $\widetilde{\rho}(r,z)$ denote the vertical velocity and the density as a function of position, respectively. The vertical fluxes of volume, mass and momentum can be used in a non-unique way to define an equivalent width b_A, an equivalent vertical velocity w_A and an equivalent density ρ_A:

$$\frac{w_A b_A^2}{A_1} = 2\int_0^\infty \widetilde{w} r \,\mathrm{d}r,$$

$$\frac{\rho_A w_A b_A^2}{A_2} = 2\int_0^\infty \widetilde{\rho}\widetilde{w} r \,\mathrm{d}r,$$

$$\frac{\rho_A w_A^2 b_A^2}{A_3} = 2\int_0^\infty \widetilde{\rho}\widetilde{w}^2 r \,\mathrm{d}r,$$

where A_1, A_2 and A_3 are constants. In this way, top-hat profiles are defined by the values that the vertical velocity and the density would have if they were uniform on a width b_T across the plume at a given height. 'Top-hat' variables are denoted by the subscript $(\cdot)_T$. The three equations above then give $A_1 = A_2 = A_3 = 1$. For Gaussian profiles $A_1 = 1$, $A_2 = 2$ and $A_3 = 3$ (i.e. when assuming $w_G = w_m \exp(-r^2/b_G^2)$ and $\rho_G = \rho_0 + (\rho_m - \rho_0)\exp(-r^2/b_G^2)$ on $0 \leqslant r \leqslant \infty$; the subscript m denoting the value on the plume centreline and the subscript $(\cdot)_G$ 'Gaussian' variables) and b, w and ρ must be replaced by b_G, w_m and ρ_m, respectively, in the left-hand side of the above equations. There is some discrepancy in the literature regarding the appropriate choice of constants A_1, A_2 and A_3 and the reader is referred to Linden (2000) for further discussion of the plume equations.

From now on, we use only top-hat quantities and drop the subscript $(\cdot)_T$. The conservation equations can then be written (see MTT):

$$\frac{\mathrm{d}(\rho w b^2)}{\mathrm{d}z} = 2bu_e\rho_0, \tag{2.1}$$

$$\frac{\mathrm{d}(\rho w^2 b^2)}{\mathrm{d}z} = -gb^2\rho', \tag{2.2}$$

$$\frac{\mathrm{d}(wb^2)}{\mathrm{d}z} = 2bu_e, \tag{2.3}$$

where u_e denotes the inflow (or entrainment) velocity at the plume edge, $\rho' = \rho - \rho_0$ and ρ_0 is the density of the ambient.

2.2. *Similarity solutions*

The non-Boussinesq plume rising from a point source in a still and uniform ambient has been examined by RL who develop similarity solutions based on self-similar top-hat profiles in the plume at all heights. RL then use their similarity solutions ((2.6)–(2.8) below) to derive an expression for the entrainment velocity across the plume edge, namely

$$u_e = \alpha w\sqrt{\rho/\rho_0}, \tag{2.4}$$

where α is the entrainment coefficient. This form of entrainment, dependent not only on a characteristic vertical velocity but also on the square root of local density contrast, is the same as that suggested by Ricou & Spalding (1961) based on their empirical measurements for an arbitrary density ratio. Morton (1965) provides additional justification for this form of the entrainment on dimensional grounds assuming that the rate of entrainment into a strongly-buoyant plume is a function of ρ/ρ_0 and the local Reynolds stresses ($\propto \rho w^2$).

In contrast, for the Boussinesq case, the entrainment coefficient $\alpha_b = u_e/w$ is usually assumed constant for all heights (e.g. MTT; Turner 1986), although more complex so-called 'modified' entrainment functions in which α_b varies according to the local fluxes in the plume have been proposed (e.g. Ricou & Spalding 1961; Kotsovinos & List 1977). Hunt & Kaye (2005) showed that some reduced entrainment Boussinesq flows can be modelled using a constant α formulation. There is evidence to suggest that entrainment into a non-Boussinesq plume is not constant (Cetegen, Zukoski & Kubota 1984) and in the model above, $u_e/w = \alpha\sqrt{\rho/\rho_0}$ is dependent on the local density ratio between the plume and the environment which, in turn, depends on the local fluxes $Q(z)$, $B(z)$ and $M(z)$. RL's similarity solutions with this form of entrainment show good agreement with available data and in the interests of developing a simplified analytical model for the virtual origin of the non-Boussinesq area plume we shall adopt this form herein.

Combining (2.3) and (2.1) yields the conservation of the flux of density deficit $(\rho - \rho_0)wb^2$. This is equivalent to the conservation of the quantity

$$B = -\pi g\frac{\rho - \rho_0}{\rho_0}wb^2, \tag{2.5}$$

which has the dimensions of buoyancy flux $\mathrm{m}^4\,\mathrm{s}^{-3}$ (see RL). RL then seek a solution of (2.1), (2.2) and (2.3) in a similarity form, and, using the fact that when $\rho' \ll \rho_0$ the Boussinesq form must be recovered, they conclude that in the general non-Boussinesq case:

$$w = C_1\pi^{-1/3}B^{1/3}z^{-1/3}, \tag{2.6}$$

$$b = C_2(\rho/\rho_0)^{-1/2} z, \tag{2.7}$$

$$g\frac{\rho'}{\rho} = -C_3 \pi^{-2/3} B^{2/3} z^{-5/3}, \tag{2.8}$$

(in the present work, the signs are set for a light plume rising in a denser environment) and show that

$$u_e = \tfrac{5}{6} C_2 w \sqrt{\rho/\rho_0},$$

where C_1, C_2 and C_3 are constants. Using (2.4), the unknown constants are expressed in terms of α only:

$$C_1 = \left(\tfrac{3}{4}\right)^{1/3} \left(\tfrac{6}{5}\alpha\right)^{-2/3}, \quad C_2 = \tfrac{6}{5}\alpha, \quad C_3 = \left(\tfrac{4}{3}\right)^{1/3} \left(\tfrac{6}{5}\alpha\right)^{-4/3},$$

and $C_1 C_2^2 C_3 = 1$. In the case when $\rho' \ll \rho_0$, B denotes the buoyancy flux and the results of MTT for the Boussinesq limit are recovered.

2.3. *Non-dimensional form of the equations*

A characteristic length scale ℓ, formed from the flux B and g, is

$$\ell = \frac{B^{2/5}}{(\kappa g)^{3/5}} \quad \text{where} \quad \kappa = \frac{3^{5/3}(2\pi)^{2/3}}{5^{4/3}} \alpha^{4/3}.$$

The non-dimensional constant κ is chosen so that the non-dimensional form of the equations is as simple as possible. For $\alpha = 0.1$, we have $\kappa \approx 0.1154$. Using the notation

$$x = \frac{\rho}{\rho_0}, \quad M = x\pi w^2 b^2, \quad Q = \pi w b^2, \quad B = \pi g(1-x) w b^2,$$

(note that $b = \pi^{-1/2} Q (x/M)^{1/2}$ and $w = M/(xQ)$), we define the non-dimensional height ζ, volume flux $\mathscr{Q}$ and momentum flux $\mathscr{M}$ such that

$$\zeta = z/\ell,$$

$$\mathscr{Q} = \frac{gQ}{\kappa^{3/2} g^{1/2} \ell^{5/2}},$$

$$\mathscr{M} = \frac{M}{\kappa^{3/2} g \ell^3}.$$

The length ℓ is the height below which the non-Boussinesq effects are dominant. For $z \gg \ell$, the Boussinesq limit of the plume is recovered. This length scale is discussed further in Woods (1997). The conservation equations now reduce to the dimensionless form:

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}\zeta} = \frac{4}{3} \frac{\mathscr{Q}}{\mathscr{M}} \left(1 - \frac{1}{\mathscr{Q}}\right),$$

$$\frac{\mathrm{d}\mathscr{Q}}{\mathrm{d}\zeta} = \frac{5}{3} \mathscr{M}^{1/2},$$

and $x = 1 - 1/\mathscr{Q}$; for a light rising plume in a denser environment, x increases from a small value to 1. With the present notation, RL's solution ((2.6), (2.7), (2.8)) can be written:

$$\mathscr{M} = \zeta^{4/3}, \tag{2.9}$$

$$\mathscr{Q} = 1 + \zeta^{5/3}, \tag{2.10}$$

$$x = \frac{1}{1 + \zeta^{-5/3}}. \tag{2.11}$$

In the Boussinesq limit, i.e. for $\zeta \gg 1$, it yields: $\mathscr{M} = \zeta^{4/3}$, $\mathscr{Q} = \zeta^{5/3}$ and $x = 1 - \zeta^{-5/3}$.

2.4. *Mass flux in non-Boussinesq plumes*

We can define the mass flux, expressed in buoyancy units (i.e. $m^4\,s^{-3}$), as

$$G = gx\pi w b^2 = gxQ.$$

The non-dimensional mass flux is then:

$$\mathscr{G} = \frac{G}{(\kappa g)^{3/2}\ell^{5/2}} = \frac{G}{B},$$

and we have:

$$\mathscr{G} = \zeta^{5/3}.$$

Therefore, although the volume flux may, in general, be non-zero at the origin, the mass flux has a zero value at the origin. The governing equations can now be rewritten as:

$$\frac{\mathrm{d}M}{\mathrm{d}z} = \frac{BG}{gM}, \quad \frac{\mathrm{d}G}{\mathrm{d}z} = 2\pi^{1/2}\alpha g\sqrt{M}, \quad x = \frac{G}{G+B},$$

and these can then be expressed in the following convenient non-dimensional form in terms of mass and momentum flux only:

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}\zeta} = \tfrac{4}{3}\frac{\mathscr{G}}{\mathscr{M}}, \tag{2.12}$$

$$\frac{\mathrm{d}\mathscr{G}}{\mathrm{d}\zeta} = \tfrac{5}{3}\sqrt{\mathscr{M}}, \tag{2.13}$$

$$x = \frac{\mathscr{G}}{1+\mathscr{G}}. \tag{2.14}$$

The system of equations (2.12)–(2.14) has the self-similar solution $\mathscr{M}=\zeta^{4/3}$, $\mathscr{G}=\zeta^{5/3}$ and $x=(1+\zeta^{-5/3})^{-1}$. These solutions are identical to (2.9)–(2.11). However, in (2.10), we note that $\mathscr{Q}\to 1$ as $\zeta\to 0$ which is non-intuitive, whereas here, $\mathscr{G}$ has the intuitive limit $\mathscr{G}\to 0$ as $\zeta\to 0$. This solution is very similar to the classical Boussinesq formulation, except that $\mathscr{G}$ replaces $\mathscr{Q}$. Note that in the Boussinesq limit, $G\approx gQ$, hence $\mathscr{G}\approx\mathscr{Q}$. Thus, using either of these two quantities in the Boussinesq case is the same, while the quantity with a sound physical meaning in the non-Boussinesq case is the mass flux. In other words, most Boussinesq results can be transposed directly to the non-Boussinesq case provided care is taken to use the mass flux instead of the volume flux.

3. Pure, lazy and forced plumes

3.1. *Non-dimensional formulation*

In (2.13), the right-hand side is always positive. Therefore, we can use $\mathscr{G}$ as the main variable, writing:

$$\frac{\partial\mathscr{M}}{\partial\mathscr{G}} = \frac{\mathrm{d}\mathscr{M}}{\mathrm{d}\zeta}\frac{\partial\zeta}{\partial\mathscr{G}}.$$

From (2.12) and (2.13), $4\mathscr{G}d\mathscr{G} = 5\mathscr{M}^{3/2}d\mathscr{M}$, which on integration gives:

$$\mathscr{G}^2 - \mathscr{G}_h^2 = \mathscr{M}^{5/2} - \mathscr{M}_h^{5/2}, \tag{3.1}$$

where $\mathscr{G}_h$ and $\mathscr{M}_h$ are the values of $\mathscr{G}$ and $\mathscr{M}$ at a height $\zeta_h = z_h/\ell$ (when conditions are known at the actual source, substitute $\zeta_h = 0$ throughout the analysis that follows).

We define the non-Boussinesq source parameter as

$$\Gamma_{nb} = \frac{\mathscr{G}_h^2}{\mathscr{M}_h^{5/2}},$$

and substitution of the solutions for B, G and M, from (2.6)–(2.8), valid for a plume issuing from a point source, yields $\Gamma_{nb} = 1$. This so-called 'pure' plume source may also arise from an area source providing the source fluxes satisfy $\Gamma_{nb} = 1$. If the source has a deficit of momentum compared with the aforementioned pure plume, then $\Gamma_{nb} > 1$ and the resulting plume is referred to as a 'lazy' plume. In contrast, if the source has an excess of momentum compared with the pure plume, then $\Gamma_{nb} < 1$ and the resulting plume is 'forced'.

With this notation, (3.1) may be written

$$\mathscr{G}^2 - \mathscr{M}^{5/2} = \mathscr{M}_h^{5/2}(\Gamma_{nb} - 1) \quad \text{or} \quad \mathscr{G}^2 - \mathscr{M}^{5/2} = \mathscr{G}_h^2(1 - 1/\Gamma_{nb}),$$

and (2.12) and (2.13), respectively, become

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}\zeta} = \frac{4}{3}\,\frac{\sqrt{\mathscr{M}_h^{5/2}(\Gamma_{nb} - 1) + \mathscr{M}^{5/2}}}{\mathscr{M}}, \tag{3.2}$$

$$\frac{\mathrm{d}\mathscr{G}}{\mathrm{d}\zeta} = \tfrac{5}{3}(\mathscr{G}^2 - \mathscr{G}_h^2(1 - 1/\Gamma_{nb}))^{1/5}. \tag{3.3}$$

To solve completely for the flow in the plume, one of these differential equations has to be integrated. Being the simpler of the two, (3.3) will be solved in the following sections.

3.2. *Jet and source lengths*

In dimensional variables, the parameter Γ_{nb} takes the form

$$\Gamma_{nb} = \frac{5}{8\alpha\sqrt{\pi}g^2}\,\frac{G_h^2 B_h}{M_h^{5/2}}.$$

This is similar to the equivalent expression for the Boussinesq case (see Morton 1959; Hunt & Kaye 2001, 2005). By analogy with the Boussinesq parameter Γ_b, we note that the parameter Γ_{nb} represents the ratio of two length scales, namely, the non-Boussinesq jet-length L_{jnb} and the non-Boussinesq source-length L_{anb} such that $L_{jnb} \sim M_h^{3/4}/B_h^{1/2}$ and $L_{anb} \sim x_h Q_h/M_h^{1/2} = G_h/(g M_h^{1/2})$. More precisely, we define:

$$L_{jnb} = \mathscr{M}_h^{3/4}\ell = \mathscr{G}^{3/5}\Gamma_{nb}^{-3/10}\ell, \quad L_{anb} = \frac{\mathscr{G}_h}{\mathscr{M}_h^{1/2}}\ell = \mathscr{G}^{3/5}\Gamma_{nb}^{1/5}\ell, \tag{3.4}$$

and therefore,

$$\Gamma_{nb} = \left(\frac{L_{anb}}{L_{jnb}}\right)^2.$$

Further algebra shows that:

$$L_{jnb} = \frac{\sqrt{10}}{3\pi^{1/4}\alpha^{1/2}}\frac{M_h^{3/4}}{B_h^{1/2}}, \quad L_{anb} = \frac{5}{6\alpha\sqrt{\pi}}\frac{G_h}{g M_h^{1/2}}.$$

Owing to the definitions (3.4) of L_{jnb} and L_{anb}, which use only the non-Boussinesq length and non-dimensional variables, the constants in the above expressions for L_{jnb} and L_{anb} are uniquely prescribed. This then permits direct and quantitative comparisons of all length scales rather than merely order of magnitude comparisons. Note also that these length scales can be rewritten as $L_{jnb}=\mathscr{G}^{3/5}\Gamma_{nb}^{-3/10}\ell$ and $L_{anb}=\mathscr{G}^{3/5}\Gamma_{nb}^{1/5}\ell$.

From a physical viewpoint, the length scale L_{jnb} provides a measure of the vertical distance over which the momentum flux developed by the action of the buoyancy force is comparable with the source momentum flux and, thereafter, the flow is more plume-like than jet-like. The length scale L_{anb} is a measure of the horizontal extent of the source and the distance from a pure-plume distributed source to its virtual origin.

3.3. *Virtual origin of pure plumes* $\Gamma_{nb}=1$

With $\Gamma_{nb}=1$, (3.1) and (3.3) may be solved to give

$$\mathscr{G}=\left(\mathscr{G}_h^{3/5}+\zeta-\zeta_h\right)^{5/3},\quad \mathscr{M}=\left(\mathscr{G}_h^{3/5}+\zeta-\zeta_h\right)^{4/3}.$$

By choosing the origin of the vertical axis such that $\mathscr{G}_h^{3/5}-\zeta_h=0$, which can always be achieved by a change of origin, the self-similar solution ($\mathscr{G}=\zeta^{5/3}$, $\mathscr{M}=\zeta^{4/3}$) is recovered. The virtual origin of the equivalent source $(B_h, 0, 0)$ is therefore located at:

$$\zeta_v=\zeta_h-\mathscr{G}_h^{3/5}. \tag{3.5}$$

In dimensional variables, this is $z_v=z_h-L_{jnb}$ with $L_{anb}=L_{jnb}$.

The virtual origin of a pure plume can be determined in the single step shown above, however, for non-pure-plume source conditions, two steps are required (cf. the forced Boussinesq plumes studied by Morton 1959). The first step is exact and replaces the actual source $(B_0, G_h, M_h, z=z_h)$ with an area source $(B_0, G_\ell, 0, z=z_\ell)$ for lazy plumes, and a point source $(B_0, 0, M_f, z=z_f)$ for forced plumes. The second step is asymptotic and leads us to the virtual point source $(B_0, 0, 0, z=z_v)$.

4. Lazy plumes $\Gamma_{nb}>1$

4.1. *Integration of the lazy plume equations*

From (3.1), it follows that

$$\mathscr{G}_\ell^2=\mathscr{G}_h^2-\mathscr{M}_h^{5/2}>0, \tag{4.1}$$

thus defining the real positive constant $\mathscr{G}_\ell$. Integrating (3.3) backwards (to small ζ), it follows that for the plume considered, there exists a height ζ_ℓ such that:

$$\mathscr{G}(\zeta_\ell)=\mathscr{G}_\ell,\quad \mathscr{M}(\zeta_\ell)=0,$$

and for all heights above ζ_ℓ,

$$F_\ell(\mathscr{G}/\mathscr{G}_\ell)=\tfrac{5}{3}\mathscr{G}_\ell^{-3/5}(\zeta-\zeta_\ell)\quad\text{where}\quad F_\ell(X)=\int_1^X\frac{\mathrm{d}u}{(u^2-1)^{1/5}}. \tag{4.2}$$

Note that z_ℓ is determined by the equation

$$\zeta_h-\zeta_\ell=\tfrac{3}{5}\mathscr{G}_\ell^{3/5}F_\ell(\mathscr{G}_h/\mathscr{G}_\ell).$$

The plume equations have no solution for $\zeta\leqslant\zeta_\ell$.

4.2. *Shape of the lazy plumes near* z_ℓ

The behaviour of a lazy plume is given by:

$$\mathscr{G}=\mathscr{G}_\ell\times F_\ell^{-1}\left(\frac{5}{3}\mathscr{G}_\ell^{-3/5}(\zeta-\zeta_\ell)\right),$$
$$\mathscr{M}=\mathscr{G}_\ell^{4/5}((\mathscr{G}/\mathscr{G}_\ell)^2-1)^{2/5}.$$

For a parameter $X \to 1$, we have $F_\ell(X) \sim (5/2^{11/5})(X-1)^{4/5}$ (see Appendix A). Therefore, for a parameter $Y \to 0$, we have $F_\ell^{-1}(Y) = 1 + (2^{11/4}/5^{5/4})Y^{5/4} + O(Y^{9/4})$. It follows for $\zeta \to \zeta_\ell$:

$$\mathscr{G} \approx \mathscr{G}_\ell \times \left(1 + \frac{2^{11/4}}{3^{5/4}} \mathscr{G}_\ell^{-3/4} (\zeta - \zeta_\ell)^{5/4}\right),$$

$$\mathscr{M} \approx \frac{2^{3/2}}{3^{1/2}} \mathscr{G}_\ell^{1/2} \times (\zeta - \zeta_\ell)^{1/2}.$$

Since $x = \mathscr{G}/(1+\mathscr{G})$, $w = \rho_0 M/G$ and $b = \pi^{-1/2} g^{-1} G/(xM)^{1/2}$, it follows that, for $\zeta \to \zeta_\ell$, $x \to \mathscr{G}_\ell/(1+\mathscr{G}_\ell)$, $w \to 0$ and $b \to \infty$. As $x \to \mathscr{G}_\ell/(1+\mathscr{G}_\ell)$, then near z_ℓ, the plume width takes the form

$$b \sim \frac{3^{1/4}}{2^{3/4}\pi^{1/2}} g^{-1} \mathscr{G}_\ell^{1/4} (1+\mathscr{G}_\ell)^{1/2} \times (\zeta - \zeta_\ell)^{-1/4}.$$

This implies that asymptotically, the limit of a lazy non-Boussinesq plume requires an infinite source area in order to have a non-zero mass flux $\mathscr{G}_\ell$ with a zero vertical velocity. The decreasing plume width with ζ for ζ near ζ_ℓ is consistent with the result for Boussinesq plumes (Hunt & Kaye 2005) and indicates that in this region the entrainment is weak and that the acceleration of flow dominates the dynamics.

4.3. *Virtual origin of lazy plumes*

The study of virtual origins, i.e. the study aiming to determine which (point source) pure plume most closely approximates the flow from the actual plume, concentrates on the far field, since the further it is from the source, the closer any plume approximates to a pure plume.

For large X, it can be shown that $F_\ell(X) = 5X^{3/5}/3 + \varpi_\ell + O_{X\to\infty}(X^{-7/5})$, see Appendix A, where the constant $\varpi_\ell \approx -1.42$. Thus, for large $\zeta - \zeta_\ell$,

$$\zeta = \mathscr{G}^{3/5} + \zeta_\ell + \tfrac{3}{5}\mathscr{G}_\ell^{3/5}\varpi_\ell + O_{\mathscr{G}\to\infty}(1).$$

Therefore, the virtual origin for a lazy plume is defined by

$$\zeta_v = \zeta_\ell + \tfrac{3}{5}\mathscr{G}_\ell^{3/5}\varpi_\ell.$$

Note that since $\varpi_\ell \approx -1.42 < 0$, the virtual origin is always below the height ζ_ℓ. Note also from (3.1) that for large $\mathscr{G}$,

$$\zeta = \mathscr{M}^{3/4}\left(1 + \mathscr{G}_\ell/\mathscr{M}^{5/2}\right)^{3/10} + \zeta_v + O_{\mathscr{M}\to\infty}(1) = \mathscr{M}^{3/4} + \zeta_v + O_{\mathscr{M}\to\infty}(1)$$

which shows that the virtual origin calculated from $\mathscr{M}$ is identical to the virtual origin calculated from $\mathscr{G}$.

Furthermore, if $F_\ell(X)$ is developed in series, an expression similar to that given by Hunt & Kaye (2001) for Boussinesq plumes is obtained:

$$\zeta_v = \zeta_h + \frac{\mathscr{M}_h^{1/2}}{\mathscr{G}_h} \Gamma_{nb}^{-1/5}(1-\delta),$$

where

$$\delta = \tfrac{3}{5}\sum_{n=1}^{\infty}\left(\frac{\gamma^{-2n}}{5^{n-1} n!\,(10n-3)} \prod_{j=1}^{n}(1+5(j-1))\right)$$

and

$$\gamma = \left(\frac{|\Gamma_{nb}-1|}{\Gamma_{nb}}\right)^{-1/2}.$$

4.4. *Matched expansions for lazy plumes*

For a lazy plume, the flow above the 'equivalent' point source found with the virtual origin theory developed provides only an approximation to the flow above the actual source with the accuracy of this approximation increasing with increasing ζ. When the plume is unobstructed and free to rise, these asymptotic solutions provide a satisfactory means of modelling the flow far away from the general area source. However, when modelling the flow above area sources in confined regions, there is a need for a robust description of the plume for heights smaller than the maximum rise height, i.e. the height H of the confining region.

For large fires in enclosures, the vertical height over which the plume is free to develop may be small compared with the height required for the asymptotic solutions to hold to a satisfactory level of accuracy. The approach of replacing the actual source by an 'equivalent' point source at the virtual origin is then inappropriate and an alternative is needed. A matched expansion of the plume solution valid for both small and large ζ is now proposed to meet this need.

We seek an approximation of the function $Y = F_\ell(X)$ which attains an acceptable level of accuracy for any $X \geqslant 1$ and the inverse function $X = F_\ell^{-1}(Y)$ that is suitably accurate for any $Y \geqslant 0$. Approximations of F_ℓ and F_ℓ^{-1} are useful if they are simple and of high enough accuracy. Moreover, it is necessary in general to have a higher-accuracy approximation for F_ℓ^{-1} than F_ℓ, as for given source conditions, F_ℓ need be evaluated once only, while F_ℓ^{-1} must be evaluated at each height. Discussion in Appendix B shows that a good approximation of F_ℓ^{-1} is:

$$\breve{F}_\ell^{-1}(Y) = 1 + \left(\frac{2^{11/5}}{5}Y\right)^{5/4}\left(1 + \frac{3^4}{2^{33/5}5}Y\right)^{5/12}.$$

From the mass flux conservation and the above expression for $\breve{F}_\ell^{-1}(Y)$, a good approximation for $\zeta \geqslant 0$ of the mass and momentum fluxes in a non-Boussinesq lazy plume with general starting conditions yielding $\Gamma_{nb} > 1$ is thus:

$$\mathscr{G} = \mathscr{G}_\ell\left\{1 + \left(\frac{2^{11/5}}{5}Y\right)^{5/4}\left(1 + \frac{3^4}{2^{33/5}5}Y\right)^{5/12}\right\},$$
$$\mathscr{M} = (\mathscr{G}^2 - \mathscr{G}_\ell^2)^{2/5},$$

where,

$$Y = \tfrac{5}{3}\mathscr{G}_\ell^{-3/5}(\zeta - \zeta_\ell).$$

5. Forced plumes $\Gamma_{nb} < 1$

5.1. *Integration of the forced plume equations*

From (3.1), it follows that

$$\mathscr{M}_f^{5/2} = \mathscr{M}_h^{5/2} - \mathscr{G}_h^2 > 0, \tag{5.1}$$

thus defining the real positive constant $\mathscr{M}_f$. Integrating (3.3) backwards (to small ζ), it follows that for the plume considered, there exists a height ζ_f such that:

$$\mathscr{G}(\zeta_f) = 0, \quad \mathscr{M}(\zeta_f) = \mathscr{M}_f,$$

and for all heights above ζ_f,

$$F_f\left(\mathscr{G}/\mathscr{M}_f^{5/4}\right) = \tfrac{5}{3}\mathscr{M}_f^{-3/4}(\zeta - \zeta_f) \quad \text{where} \quad F_f(X) = \int_0^X \frac{\mathrm{d}u}{(u^2+1)^{1/5}}. \tag{5.2}$$

Note that z_f is determined by the equation

$$\zeta_h - \zeta_f = \tfrac{3}{5}\mathscr{M}_f^{3/4}\, F_f\big(\mathscr{G}_h/\mathscr{M}_f^{5/4}\big).$$

5.2. *Shape of the forced plumes near z_f*

For $X \to 0$, $F_f(X) \sim X$. Hence,

$$\mathscr{G} \sim \tfrac{5}{3}\mathscr{M}_f^{1/2}(\zeta - \zeta_f)$$

when $\zeta \to \zeta_f$. Since $\mathscr{M} = \mathscr{M}_f(1 + \mathscr{G}^2/\mathscr{M}^{5/2})^{2/5}$, it also follows that

$$\mathscr{M} - \mathscr{M}_f \sim \tfrac{10}{9}\mathscr{M}_f^{-3/2}(\zeta - \zeta_f)^2.$$

Additionally, since $x = \mathscr{G}/(1 + \mathscr{G})$, $w = \rho_0 M/G$ and $b = \pi^{-1/2} g^{-1} G/(xM)^{1/2}$, it follows that, for $\zeta \to \zeta_f$, $x \to 0$, $w \to \infty$ and $b \to 0$.

5.3. *Virtual origin of forced plumes*

For large X, it can be shown that $F_f(X) = 5X^{3/5}/3 + \varpi_f + O_{X\to\infty}(X^{-7/5})$, see Appendix A, where the constant $\varpi_f \approx -0.84$. Thus, for large $\zeta - \zeta_f$,

$$\zeta = \mathscr{G}^{3/5} + \zeta_f + \tfrac{3}{5}\mathscr{M}_f^{3/4}\varpi_f + O_{\mathscr{G}\to\infty}(1).$$

Therefore, the virtual origin for a forced plume is defined by

$$\zeta_v = \zeta_f + \tfrac{3}{5}\mathscr{M}_f^{3/4}\varpi_f.$$

Note that since $\varpi_f \approx -0.84 < 0$, the virtual origin is always below the height ζ_f.

When $1/2 < \Gamma_{nb} < 1$, a series estimate of the virtual-origin correction may be developed. The resulting series is identical to that presented by Hunt & Kaye (2001), and the number of terms required to give the solution to a prescribed level of accuracy is given therein.

5.4. *Matched expansions for forced plumes*

As for lazy plumes, a matched expansion is useful. A good approximation of F_f^{-1} is:

$$\breve{F}_f^{-1}(Y) = Y\big(1 + \big(\tfrac{3}{5}\big)^{3/2} Y\big)^{2/3},$$

see Appendix B. Our suggested approximation of the mass and momentum fluxes in a non-Boussinesq forced plume with general starting conditions yielding $\Gamma_{nb} < 1$ is thus:

$$\mathscr{G} = \mathscr{M}_f^{5/4}\, Y\big(1 + (3/5)^{3/2} Y\big)^{2/3},$$
$$\mathscr{M} = \mathscr{M}_f\big(1 + \mathscr{G}^2/\mathscr{M}_f^{5/2}\big)^{2/5},$$

where:

$$Y = \tfrac{5}{3}\mathscr{M}_f^{-3/4}(\zeta - \zeta_f).$$

5.5. *Estimate of fluxes in heated gas plumes*

As an example, we now consider how the results derived above may be applied to a heated gas plume. The source is assumed to be a circular heated surface, which may also be porous – in which case we assume that the fluid released through the surface has the same temperature T_h as the surface. The diameter of the heated surface is denoted by D_h. If the heat release rate is $\dot{Q}_c$, the buoyancy flux is:

$$B = \frac{g\dot{Q}_c}{c_p \rho_0 T_0},$$

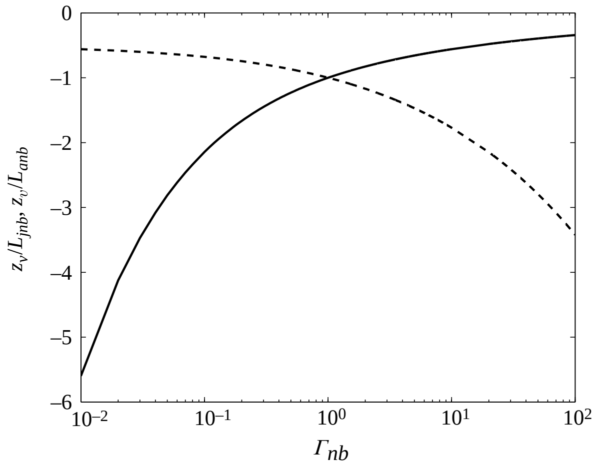

FIGURE 1. Location of the virtual origin. —, z_v/L_{anb} as a function of Γ_{nb}; - - -, z_v/L_{jnb} as a function of Γ_{nb}.

where c_p is the specific heat of the fluid, and T_0 the ambient temperature (see e.g. Delichatsios 1995).

Assuming the gas is ideal, $x_h = T_h/T_0$, and therefore:

$$G_h = \frac{x_h}{1 - x_h} B,$$
$$M_h = \frac{4G_h^2}{\pi x_h g^2 D_h^2}.$$

From the definition of Γ_{nb}, some algebra shows:

$$\Gamma_{nb} = \frac{(1 - x_h)^3}{\sqrt{x_h}} \left(\frac{5}{12\alpha} \frac{D_h}{\ell} \right)^5,$$

where, as before, $\ell = B^{2/5}/(\kappa g)^{3/5}$. Here, the heated surface is located at height $z_h = 0$. Applying the above theory on non-Boussinesq plumes the virtual origins are located as follows:

$$\text{for } \Gamma_{nb} < 1, \quad \zeta_v = -\tfrac{3}{5}\mathscr{M}_f^{3/4}\, F_f\left(\mathscr{G}_h/\mathscr{M}_f^{5/4}\right) + \tfrac{3}{5}\mathscr{M}_f^{3/4}\varpi_f,$$
$$\text{for } \Gamma_{nb} = 1, \quad \zeta_v = -\mathscr{G}_h^{3/5},$$
$$\text{for } \Gamma_{nb} > 1, \quad \zeta_v = -\tfrac{3}{5}\mathscr{G}_\ell^{3/5}\, F_\ell(\mathscr{G}_h/\mathscr{G}_\ell) + \tfrac{3}{5}\mathscr{G}_\ell^{3/5}\varpi_\ell.$$

Figure 1 shows the origin correction, as determined by the above equations.

6. Conclusions

The behaviour of turbulent non-Boussinesq plumes rising from horizontal area sources in a quiescent environment of uniform density has been considered. Analytical solutions of the non-Boussinesq conservation equations, expressing conservation of the fluxes of buoyancy B, momentum M and mass G, are developed for plumes with general source conditions B_0, M_0 and G_0 in which the source momentum acts in the same sense as the buoyancy force. The plume is characterized by the relative magnitudes of the fluxes at the source via the dimensionless parameter

$\Gamma_{nb} \propto B_0 G_0^2 M_0^{-5/2}$. As the plume rises, the local density contrast reduces owing to entrainment, and the conservation equations are non-dimensionalised based on the length scale ℓ that characterizes the height above the source over which non-Boussinesq effects are dominant.

Analytical solutions of the conservation equations for an area source reveal the dependence with height of the mass and momentum fluxes and the origin location of an equivalent point-source virtual plume. The location of this virtual source (the virtual origin), from which the power-law expressions of RL may be directly applied, is determined for forced plumes ($0 < \Gamma_{nb} < 1$), pure plumes ($\Gamma_{nb} = 1$) and lazy plumes ($\Gamma_{nb} > 1$). The far-field behaviour of this point source of buoyancy alone asymptotes to the far-field flow above the non-Boussinesq area source of interest. In general, the distance between the actual and virtual sources scales on ℓ; however, for highly forced $\Gamma_{nb} \ll 1$ and very lazy plumes $\Gamma_{nb} \gg 1$, the dominant length scale is the jet-length and the source diameter, respectively. The distance between the actual and the virtual source decreases as Γ_{nb} increases and in the limit of an infinitely lazy plume the actual and virtual sources are coincident. For forced plumes, our solutions reduce to those of Morton (1959) and Hunt & Kaye (2001) in the Boussinesq limit.

This knowledge of the virtual origin location provides a convenient means of approximating the far-field flow above a non-Boussinesq area source with an idealized point source and with the accuracy of the approximation increasing with increasing height. In many cases of practical interest, e.g. fires in enclosures, it is desirable to predict the flow in the plume, not only in the far field, and an approximate model for the flow at all heights has been developed by matching asymptotic solutions valid in the near- and far-field regions. This matching approach is new and appropriate data in both the near and far field of fire plumes is required before the model may be fully validated.

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Appendix A. Asymptotics of the functions F_ℓ and F_f

The plume equations can be solved for general source conditions provided the functions F_ℓ and F_f are computed. They are both defined by variable bound integrals that cannot be expressed in terms of standard functions. The aim of Appendices A and B is to gain more understanding of the behaviour of these functions in two stages. For the determination of virtual origins only the asymptotic behaviour of the functions is needed. This is examined in Appendix A. In order to find a 'matched' analytical solution of the plume equations, uniform approximations are needed; these are developed in Appendix B.

A.1. *Expansion of $F_\ell(X)$ for large X*

Our solution for the lazy plume requires evaluation of the function

$$F_\ell(X) = \int_1^X \frac{\mathrm{d}u}{(u^2-1)^{1/5}}, \tag{A 1}$$

for an upper limit of integration X that is dependent on the source conditions. For $u \gg 1$, $(u^2-1)^{-1/5} \sim u^{-2/5}$ and therefore

$$F_\ell(X) \sim \tfrac{5}{3} X^{3/5}.$$

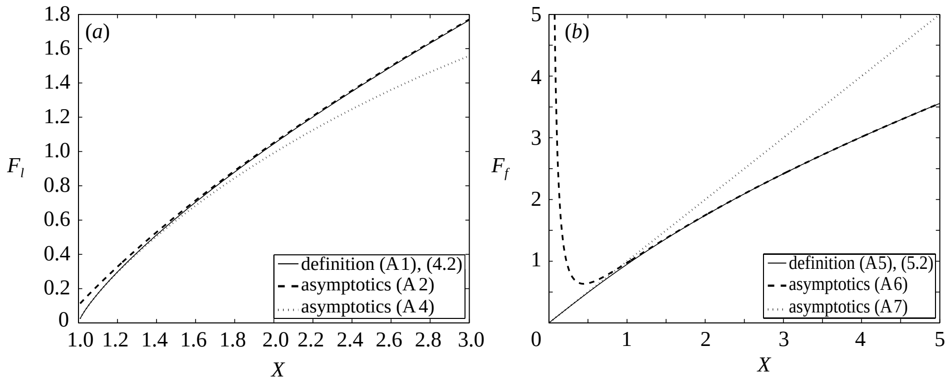

FIGURE 2. Functions F_ℓ and F_f. (*a*) For the lazy plume, the approximations to F_ℓ for large X (A2) and as $X \to 1$ (A4) are shown. (*b*) For the forced plume, the approximations to F_f for large X (A6) and as $X \to 0$ (A7) are shown.

Moreover, since

$$\left(1-(1-u^{-2})^{1/5}\right)/(u^2-1)^{1/5} \sim \tfrac{1}{5}u^{-12/5},$$

$$F_\ell(X) = \tfrac{5}{3}X^{3/5} + \varpi_\ell - \tfrac{1}{7}X^{-7/5} + O_{X\to\infty}\left(X^{-12/5}\right), \tag{A 2}$$

where the constant ϖ_ℓ is defined as

$$\varpi_\ell = \int_1^\infty \frac{1-(1-1/u^2)^{1/5}}{(u^2-1)^{1/5}}\,\mathrm{d}u - \tfrac{5}{3}.$$

The constant ϖ_ℓ may be evaluated by numerical integration giving $\varpi_\ell = -1.421038$. The same result can be found by term by term integration of the integrand expanded as a power series:

$$\int_1^X \frac{\mathrm{d}u}{(u^2-1)^{1/5}} = \tfrac{5}{3}X^{3/5} - \left(\tfrac{5}{3}+\delta_\ell\right) - \tfrac{1}{7}X^{-7/5} + O_{X\to\infty}\left(X^{-17/5}\right), \tag{A 3}$$

where $\delta_\ell = -\varpi_\ell - 5/3$

$$\delta_\ell = \sum_{n=1}^{\infty}\left(\frac{1}{5^{n-1}n!\,(3-10n)}\prod_{j=1}^{n}(1+5(j-1))\right).$$

The integration is valid for all X because of the theorem of dominated convergence.

A.2. *Expansion of $F_\ell(X)$ for $X \to 1$*

For $X \to 1$, the asymptotic problem is fully regular and:

$$F_\ell(X) = \frac{5}{2^{11/5}}(X-1)^{4/5} - \frac{1}{3^2 2^{6/5}}(X-1)^{9/5} + O_{X\to 1}\left((X-1)^{14/5}\right). \tag{A 4}$$

The function F_ℓ from (A1), and the approximations to F_ℓ for large X (A2) and as $X \to 1$ (A4) are shown in figure 2(*a*).

A.3. *Expansion of $F_f(X)$ for large X*

For forced plumes, our solution requires evaluation of the function

$$F_f(X) = \int_0^X \frac{\mathrm{d}u}{(u^2+1)^{1/5}}. \tag{A 5}$$

In a similar manner as for F_ℓ, F_f can be expanded using theorems on the expansion of integrals, which yields:

$$F_f(X) = \tfrac{5}{3}X^{3/5} + \varpi_f + \tfrac{1}{7}X^{-7/5} + O_{X\to\infty}\big(X^{-12/5}\big), \tag{A 6}$$

where the constant ϖ_f is given by

$$\varpi_f = \int_1^\infty \frac{1-(1+1/u^2)^{1/5}}{(u^2+1)^{1/5}}\mathrm{d}u - \tfrac{5}{3} + \int_0^1 \frac{\mathrm{d}u}{(u^2+1)^{1/5}} \approx -0.835266.$$

Again, ϖ_f can also be computed using a series argument, cf. Appendix A1.

A.4. *Expansion of $F_f(X)$ for $X \to 0$*

For $X \to 0$, the asymptotic problem is fully regular and:

$$F_f(X) = X - \tfrac{2}{5}X^3 + O_{X\to 0}(X^4). \tag{A 7}$$

The function F_f from (A5), and the approximations to F_f for large X (A6) and as $X \to 0$ (A7) are shown in figure 2(*b*).

Appendix B. Matched approximation of F_ℓ and F_f

B.1. *Function F_ℓ*

We seek an approximation of the function $Y = F_\ell(X)$ which meets an acceptable level of accuracy for any $X \geqslant 1$. Approximations of F_ℓ are useful if they are simple and of a sufficient accuracy. The level of accuracy required will depend on the requirements of the plume model, for example, for comparison with experimental data, the accuracy should at least match the accuracy of the measurements. In this section, we seek approximations which are uniform, in the sense that they are accurate over the complete range of validity of the functions.

It was shown in Appendix A that:

$$F_\ell(X) = \tfrac{5}{3}X^{3/5} + \varpi_\ell - \tfrac{1}{7}X^{-7/5} + O_{X\to\infty}\big(X^{-12/5}\big),$$

$$F_\ell(X) = \frac{5}{2^{11/5}}(X-1)^{4/5} - \frac{1}{3^2 2^{6/5}}(X-1)^{9/5} + O_{X\to 1}\big((X-1)^{14/5}\big).$$

We seek an approximation of the inverse function $X = F_\ell^{-1}(Y)$ that is suitably accurate for any $Y \geqslant 0$. In general, it is necessary to have a higher accuracy approximation for F_ℓ^{-1} than F_ℓ (Appendix A), because for given sources conditions, F_ℓ has to be evaluated once only, while F_ℓ^{-1} has to be evaluated at each height. The inverse functions are determined as

$$F_\ell^{-1}(Y) = \big(\tfrac{3}{5}Y\big)^{5/3} + \big(\tfrac{3}{5}\big)^{2/3}|\varpi_\ell|Y^{2/3} + O_{Y\to\infty}\big(Y^{-1/3}\big), \tag{B 1}$$

$$F_\ell^{-1}(Y) = 1 + \left(\frac{2^{11/5}}{5}Y\right)^{5/4} + O_{Y\to 0}\big(Y^{9/4}\big), \tag{B 2}$$

where F_ℓ^{-1} is defined on the interval $[0,\infty[$. A simple matching function $\breve{F}_\ell^{-1}$ approximating F_ℓ^{-1} is proposed of the form:

$$\breve{F}_\ell^{-1}(Y) = 1 + \left(\frac{2^{11/5}}{5}Y\right)^{5/4}(1+aY^b)^{5/12b}, \tag{B 3}$$

where the constant a and the exponent b are to be determined. This form for $\breve{F}_\ell^{-1}$ is chosen so as to have the correct behaviour for Y close to 0. The coefficients a and b

are determined by the analysis of the behaviour of the function at infinity. As $Y \to \infty$, equating to leading order the exponent of Y and its coefficient in (B 3) with (B 1) requires

$$b = 1, \quad a = \frac{3^4}{2^{33/5}5}.$$

Thus, our approximation is

$$\breve{F}_\ell^{-1}(Y) = 1 + \left(\frac{2^{11/5}}{5}Y\right)^{5/4}\left(1 + \frac{3^4}{2^{33/5}5}Y\right)^{5/12}. \tag{B 4}$$

The respective behaviour of this approximation as $Y \to 0$ and $Y \to \infty$ is

$$\breve{F}_\ell^{-1}(Y) = 1 + \left(\frac{2^{11/5}}{5}Y\right)^{5/4} + O_{Y\to 0}\left(Y^{9/4}\right) \tag{B 5}$$

and

$$\breve{F}_\ell^{-1}(Y) = \left(\frac{3}{5}Y\right)^{5/3} + \frac{5^{1/3}2^{23/5}}{3^{10/3}}Y^{2/3} + O_{Y\to\infty}\left(Y^{-1/3}\right). \tag{B 6}$$

The function $\breve{F}_\ell^{-1}$ provides a good approximation to F_ℓ^{-1} with a maximum relative error of less than 2 % over the range $0 \leqslant Y \leqslant 100$.

B.2. *Function F_f*

Again using the results of Appendix A,

$$F_f(X) = X - \tfrac{2}{5}X^3 + O_{X\to 0}(X^4),$$
$$F_f(X) = \tfrac{5}{3}X^{3/5} + \varpi_f + \tfrac{1}{7}X^{-7/5} + O_{X\to\infty}\left(X^{-12/5}\right),$$

with the inverse functions:

$$F_f^{-1}(Y) = Y + \tfrac{2}{5}Y^3 + O_{Y\to 0}(Y^5),$$
$$F_f^{-1}(Y) = \left(\tfrac{3}{5}Y\right)^{5/3} + \left(\tfrac{3}{5}\right)^{2/3}|\varpi_f|Y^{2/3} + O_{Y\to\infty}\left(Y^{-1/3}\right),$$

where F_f^{-1} is defined on the interval $[\varpi_0, \infty[$. A simple matching function $\breve{F}_f^{-1}$ approximating F_f^{-1} is proposed of the form:

$$\breve{F}_f^{-1}(Y) = Y(1 + a_f Y)^{2/3}, \tag{B 7}$$

where the constant a_f is to be determined. As $Y \to \infty$, the matching requires

$$a_f = \left(\tfrac{3}{5}\right)^{3/2}.$$

Thus, our approximation is

$$\breve{F}_f^{-1}(Y) = \left(\tfrac{3}{5}Y\right)^{5/3}\left(1 + \frac{(5/3)^{5/2}}{Y}\right)^{2/3}. \tag{B 8}$$

The respective behaviour of this approximation as $Y \to 0$ and $Y \to \infty$ is

$$\breve{F}_f^{-1}(Y) = Y + \tfrac{2}{3}\left(\tfrac{3}{5}\right)^{5/2}Y^2 + O_{Y\to 0}(Y^3)$$

and

$$\breve{F}_f^{-1}(Y) = \left(\tfrac{3}{5}Y\right)^{5/3} + \left(\tfrac{3}{5}\right)^{5/3}\frac{5^{5/2}2}{3^{7/2}}\,Y^{2/3} + O_{Y\to\infty}(Y)^{-1/3}.$$

The function $\check{F}_f^{-1}$ provides a reasonable approximation to F_f^{-1} over the range $0 \leqslant Y \leqslant 100$ with a maximum relative error of approximately 13 %.

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